

New Improved Massive Gravity And Three Dimensional Spacetimes Of Constant Curvature And Constant Torsion

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Abstract

We derive the field equations for topologically massive gravity coupled with the most general quadratic curvature terms using the language of exterior differential forms and a first order constrained variational principle. We find variational field equations both in the presence and absence of torsion. We then show that spaces of constant negative curvature (i.e. the anti de-Sitter space AdS_3) and constant torsion provide exact solutions.

Keywords: Topologically massive gravity. Minimal massive gravity. 3D spaces of constant curvature and constant torsion.

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1 Introduction

It is often useful to study lower dimensional field theoretic models to gain further insight into fundamental interactions of nature. In particular, gravity in (1+2)-dimensions has received a lot of attention as a theoretical tool that highlights the topological aspects of gravitation. Basic questions such as whether if gravitational interactions may have a finite range [1], or in which sense a quantum gravity might be useful [2] led to insightful answers with this approach. In fact it is well known that Einstein's gravity in (1+2)-dimensions has no dynamics on its own[3]. One may introduce gravitational degrees of freedom that propagate, either by elevating the gravitational field equations to third order by including in the action a topological Chern-Simons term [4, 5], or by coupling other fields such as a dilaton scalar [6] or a gravitino field [7] to gravity. Topologically massive gravity (TMG) proved to be interesting since it admits a stationary, circularly symmetric solution that is asymptotically AdS_3 , and behave as if it is a rotating black hole [8, 9]. In many respects, BTZ solution of TMG is the analog of Kerr solution in (1+3)-dimensions. The construction of conserved quantities associated with the BTZ solution [10, 11] and the study of hidden dualities [12, 13] prove to be challenging problems in their own right. More recently, unitary extensions of TMG were sought by the addition of quadratic curvature terms to the action, thus raising the order of the Einstein field equations from three in TMG to four [14, 15, 16]. A remarkable extension of TMG, that is called New Massive Gravity (NMG) in the literature [17, 18] consists of discarding the Chern-Simons term altogether in the action, so that there are no third derivatives left in the field equations, and replacing it by a particular combination of quadratic curvature invariants that leads to fourth order field equations, but with unitarity guaranteed at least at the linearized approximation. It should be remarked that, all the models of 3D gravity discussed upto this point involve (pseudo)-Riemannian space-times. A Minimal Massive Gravity (MMG) was introduced a couple of years ago for which the variation of the action is done under a non-linear constraint that induces a dynamical space-time torsion [19, 20, 21]. Several aspects of MMG such as unitarity [22, 23], or its conserved quantities [24] and exact solutions [25, 26] have been the subject of very recent studies.

Here in what follows, the field equations for the Einstein-Chern-Simons gravity coupled with the most general quadratic curvature terms in the action are derived by a first order constrained variational principle. We make extensive use of the concise language of exterior differential forms. Variational field equations both in the presence and absence of torsion are determined. The

notion of the torsion of a material continuum has been first introduced by É. Cartan in 1922[27]. This idea found important physical applications, on the one hand, within the context of modified theories of gravity[28], while on the other hand within the context of gauge theories of continuum dislocation and disclination defects[29]. Here we will be discussing 3-dimensional space-times of constant negative curvature (i.e. AdS_3) and constant torsion[30, 31] as exact background solutions. The concept of 3-dimensional spaces of constant torsion was implicit in Cartan's work and it is called Cartan's spiral staircase in a recent review paper[29].

Notation and Conventions:

Throughout our work, we will be using the language of exterior differential forms. The metric tensor of space-time, given by $g = \eta_{ab}e^a \otimes e^b$ where $\eta_{ab} = g(X_a, X_b) = \text{diag}(-, +, +)$ is written in terms of an orthonormal basis of frame vectors $\{X_a\}$ that are dual to the co-frame 1-forms $\{e^a\}$ so that $e^a(X_b) = \delta_b^a$. $\iota_a = \iota_{X_a}$ stands for the interior product operators with respect to frame vectors X_a . $*$: $\Lambda^p(M) \rightarrow \Lambda^{3-p}(M)$ denotes the Hodge duality operator acting on p -forms. The space-time orientation is fixed with the choice of the volume 3-form $*1 = e^0 \wedge e^1 \wedge e^2$. For convenience, the following abbreviation for the exterior products $e^{ab\dots} = e^a \wedge e^b \wedge \dots$ is going to be used. A linear connection on space-time will be specified by a set of connection 1-forms $\{\omega^a_b\}$. We will work with a connection that is compatible with the metric but need not be torsion-free. Then the index raising and lowering operations commute with the covariant exterior derivatation and we have $D(\omega)\eta_{ab} = \omega_{ab} + \omega_{ba} = 0$. We specify the torsion 2-forms T^a of space-time through the first set of Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = T^a, \quad (1.1)$$

while the curvature 2-forms $R^a_b(\omega)$ through the second set of Cartan structure equations

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b(\omega). \quad (1.2)$$

The following Bianchi identities are obtained as integrability conditions of the above Cartan structure equations:

$$D(\omega)T^a = R^a_b(\omega) \wedge e^b, \quad D(\omega)R^a_b(\omega) = 0. \quad (1.3)$$

It is convenient to define contortion 1-forms K^a_b as the difference between our non-Riemannian connection 1-forms and the Riemannian (Levi-Civita) connection 1-forms $\{\hat{\omega}^a_b\}$ that satisfy the structure equations

$$de^a + \hat{\omega}^a_b \wedge e^b = 0. \quad (1.4)$$

Thus we have

$$K^a_b = \omega^a_b - \hat{\omega}^a_b \quad (1.5)$$

which are in one to one correspondence with the torsion 2-forms T^a through the structure equations

$$K^a_b \wedge e^b = T^a, \quad (1.6)$$

or conversely

$$2K_{ab} = \iota_a T_b - \iota_b T_a - e^c \iota_{ab} T_c. \quad (1.7)$$

It is not difficult to find a relation between the non-Riemannian curvature 2-forms $R^a_b(\omega)$ and the Riemannian curvature 2-forms $R^a_b(\hat{\omega})$ of the Levi-Civita connection as

$$R^a_b(\omega) = R^a_b(\hat{\omega}) + D(\hat{\omega})K^a_b + K^a_c \wedge K^c_b \quad (1.8)$$

where $D(\hat{\omega})$ denotes the covariant exterior derivative with respect to the Levi-Civita connection. The Ricci 1-forms $Ric_a = \mathcal{R}_{ab}e^b$ and the curvature scalar \mathcal{R} are obtained by contractions of the curvature 2-forms as follows:

$$Ric_a = \iota^b R_{ba}, \quad \mathcal{R} = \iota^a Ric_a = \iota^{ab} R_{ba}. \quad (1.9)$$

Moreover, the Einstein 2-forms of our non-Riemannian connection are defined by

$$G_a(\omega) = G_{ab} * e^b = *Ric_a - \frac{1}{2}\mathcal{R} * e_a = -\frac{1}{2}R^{bc}(\omega) * e_{abc}. \quad (1.10)$$

We note that in 3-dimensions, the curvature 2-forms are in one to one correspondence with the Einstein 2-forms since

$$\epsilon_{abc} G^c(\omega) = R_{ab}(\omega) \quad (1.11)$$

where ϵ_{abc} denotes the completely anti-symmetric Levi-Civita symbol in three dimensions with $\epsilon_{012} = 1$. We may therefore give the curvature 2-forms R_{ab} in 3-dimensions in terms of the Ricci 1-forms Ric_a and the curvature scalar \mathcal{R} :

$$R_{ab} = \epsilon^c_{ab} * Ric_c + \frac{1}{2}\mathcal{R}e_a \wedge e_b. \quad (1.12)$$

As a consequence, quadratic curvature invariants in 3-dimensions are related to each other through the identity

$$R^{ab} \wedge *R_{ab} - 2Ric^a \wedge *Ric_a + \frac{1}{2}\mathcal{R}^2 * 1 = 0, \quad (1.13)$$

that is, any one of the quadratic curvature invariants in 3-dimensions can be expressed in terms of the other two.

2 Action

The field equations of our model will be determined by the constrained variations of an action integral

$$I[e^a, \omega_b^a, \lambda_a] = \int_M \mathcal{L} \quad (2.1)$$

where M is a compact region on some chart on a (1+2)-dimensional Riemann-Cartan manifold. The independent variables on which the action depends are the co-frame 1-forms $\{e^a\}$, connection 1-forms $\{\omega_b^a\}$, and Lagrange multiplier 1-forms $\{\lambda_a\}$. We consider a Lagrangian density 3-form

$$\mathcal{L} = \mathcal{L}_{TMG} + \mathcal{L}_2 + \mathcal{L}_C \quad (2.2)$$

where

$$\mathcal{L}_{TMG} = \frac{1}{\mu}(\omega_b^a \wedge d\omega_a^b + \frac{2}{3}\omega_b^a \wedge \omega_c^b \wedge \omega_a^c) + \frac{1}{2K}R^{ab} \wedge *e_{ab} + \Lambda *1 \quad (2.3)$$

is the Lagrangian density of the topologically massive gravity (TMG);

$$\mathcal{L}_2 = \alpha R^{ab} \wedge *R_{ab} + \beta Ric^a \wedge *Ric_a + \gamma \mathcal{R}^2 *1 \quad (2.4)$$

is a generic quadratic curvature term with three coupling constants α , β , and γ . It should be remarked that, there are alternative ways of specifying a generic quadratic curvature invariant in three dimensions. Due to the identity (1.13), either one of the terms $R^{ab} \wedge *R_{ab}$, $Ric^a \wedge *Ric_a$ or $\mathcal{R}^2 *1$ may be made redundant in favor of others. Therefore, still keeping the coupling constants α , β and γ , we may consider without loss of generality, any one of the following combinations:

$$\begin{aligned} \mathcal{L}_2 &= (2\alpha + \beta)Ric^a \wedge *Ric_a + (\gamma - \frac{\alpha}{2})\mathcal{R}^2 *1 \\ \mathcal{L}_2 &= (\alpha + \frac{\beta}{2})R^{ab} \wedge *R_{ab} + (\gamma + \frac{\beta}{4})\mathcal{R}^2 *1 \\ \mathcal{L}_2 &= (\alpha - 2\gamma)R^{ab} \wedge *R_{ab} + (\beta + 4\gamma)Ric^a \wedge *Ric_a. \end{aligned} \quad (2.5)$$

For technical ease. we prefer to work with the second alternative. Finally,

$$\mathcal{L}_C = T^a \wedge \lambda_a + \frac{\nu}{2}\lambda_a \wedge \lambda_b \wedge *e^{ab} \quad (2.6)$$

is the constraint Lagrangian density 3-form, which in case $\nu = 0$ imposes the constraint that the connection is the torsion-free Levi-Civita connection. On the other hand if $\nu \neq 0$, the torsion 2-forms would be related with the Lagrange multiplier 1-forms in a non-trivial way. All the previously studied models such as TMG, NMG or MMG are covered as sub-cases with the choice (2.2) of the action .

3 Variational Field Equations

We evaluate the variational derivative of the total Lagrangian and find (upto a closed form)

$$\begin{aligned}
\dot{\mathcal{L}} = \dot{e}^a \wedge & \left\{ \frac{1}{2K} R^{bc} \wedge *e_{abc} + \Lambda *e_a + \left(\alpha + \frac{\beta}{2} \right) \left(\iota_a R^{bc} \wedge *R_{bc} - R^{bc} \wedge \iota_a *R_{bc} \right) \right. \\
& + \left(\gamma + \frac{\beta}{4} \right) \left(2\mathcal{R} R^{bc} \wedge *e_{abc} - \mathcal{R}^2 *e_a \right) \\
& + D(\omega) \lambda_a + \frac{\nu}{2} \lambda^b \wedge \lambda^c \wedge *e_{abc} \Big\} \\
& + \dot{\omega}^{ab} \wedge \left\{ \frac{2}{\mu} R_{ba} + \frac{1}{2K} T^c \wedge *e_{abc} + (2\alpha + \beta) D(\omega) *R_{ab} \right. \\
& + (2\gamma + \frac{\beta}{2}) D(\omega) (\mathcal{R} *e_{ab}) + \frac{1}{2} (e_b \wedge \lambda_a - e_a \wedge \lambda_b) \Big\} \\
& + \dot{\lambda}_a \wedge \left\{ T^a + \nu \lambda_b \wedge *e^{ab} \right\}. \tag{3.1}
\end{aligned}$$

Here a dot over a field variable denotes the variation of the corresponding field. We first impose the constraint

$$T^a = -\nu \lambda_b \wedge *e^{ab} \iff K_{ab} = \nu \epsilon_{abc} \lambda^c. \tag{3.2}$$

Then we go to connection variation equations which now read

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = Q^{-1} \Sigma_{ab} \tag{3.3}$$

where we set

$$Q = \frac{1}{2} - \frac{\nu}{2K} - \nu(2\gamma + \frac{\beta}{2})\mathcal{R}, \tag{3.4}$$

and

$$\Sigma_{ab} = -\frac{2}{\mu} R_{ab} + (2\alpha + \beta) D(\omega) *R_{ab} + (2\gamma + \frac{\beta}{2}) d\mathcal{R} \wedge *e_{ab}. \tag{3.5}$$

We solve (3.3) algebraically for the Lagrange multiplier 1-forms and determine

$$\lambda_a = Q^{-1} \left(-\frac{2}{\mu} Y_a + (2\alpha + \beta) W_a + (2\gamma + \frac{\beta}{2}) (\iota_a *d\mathcal{R}) \right) \tag{3.6}$$

where we introduced further abbreviations

$$Y_a = Ric_a - \frac{1}{4} e_a \mathcal{R}, \quad W_a = \iota^b (D(\omega) *R_{ba}) - \frac{1}{4} e_a (\iota^b \iota^c D(\omega) *R_{cb}). \tag{3.7}$$

Finally we substitute (3.6) into the co-frame variation equations and arrive at the Einstein field equations given as follows:

$$\begin{aligned} & \left(\frac{1}{2K} + (2\gamma + \frac{\beta}{2})\mathcal{R} \right) R^{bc} \wedge *e_{abc} + \left(\Lambda - (\gamma + \frac{\beta}{4})\mathcal{R}^2 \right) *e_a \\ & + (\alpha + \frac{\beta}{2}) \left(\iota_a R^{bc} \wedge *R_{bc} - R^{bc} \wedge \iota_a *R_{bc} \right) \\ & D\lambda_a + \frac{\nu}{2} \lambda^b \wedge \lambda^c *e_{abc} = 0. \end{aligned} \quad (3.8)$$

We note that these equations include among other terms, the Cotton-Schouten 2-forms

$$C_a \equiv D(\omega)Y_a = D(\omega)(Ric_a - \frac{1}{4}\mathcal{R}e_a) \quad (3.9)$$

that involve third derivatives of the metric components and the 2-forms

$$D_a \equiv D(\omega)W_a = D(\omega)(\iota^b(D(\omega)*R_{ba}) - \frac{1}{4}e_a\iota^b\iota^c(D(\omega)*R_{cb})) \quad (3.10)$$

that involve fourth derivatives of the metric components. We also note that Einstein field equations in the case of zero-torsion ($\nu = 0$) are given by

$$\begin{aligned} & \left(\frac{1}{2K} + (2\gamma + \frac{\beta}{2})\hat{\mathcal{R}} \right) \hat{R}^{bc} \wedge *e_{abc} + \left(\Lambda - (\gamma + \frac{\beta}{4})\hat{\mathcal{R}}^2 \right) *e_a \\ & + (\alpha + \frac{\beta}{2}) \left(\iota_a \hat{R}^{bc} \wedge *\hat{R}_{bc} - \hat{R}^{bc} \wedge \iota_a *\hat{R}_{bc} \right) \\ & - \frac{4}{\mu} \hat{C}_a + (4\alpha + 2\beta)\hat{D}_a + (4\gamma + \beta)D(\hat{\omega})(\iota_a *d\hat{\mathcal{R}}) = 0. \end{aligned} \quad (3.11)$$

Field equations (3.11) go down consistently to the Topologically Massive Gravity (TMG) field equations if the quadratic curvature terms are absent, i.e. if we set $\alpha = \beta = \gamma = 0$ above. Finally we re-write Einstein field equations in two special cases of recent interest:

New Massive Gravity (NMG): $\frac{1}{\mu} \rightarrow 0, \Lambda = 0, \alpha = 0, \beta = 1, \gamma = -\frac{3}{8}, \nu = 0$.

$$\begin{aligned} & \left(\frac{1}{2K} - \frac{1}{4}\hat{\mathcal{R}} \right) \hat{R}^{bc} \wedge *e_{abc} + \frac{1}{8}\hat{\mathcal{R}}^2 *e_a \\ & + \frac{1}{2} \left(\iota_a \hat{R}^{bc} \wedge *\hat{R}_{bc} - \hat{R}^{bc} \wedge \iota_a *\hat{R}_{bc} \right) + 2\hat{D}_a - \frac{1}{2}D(\hat{\omega})(\iota_a *d\hat{\mathcal{R}}) = 0. \end{aligned} \quad (3.12)$$

Minimal Massive Gravity (MMG): $\alpha = \beta = \gamma = 0, \nu \neq 0$.

$$\begin{aligned} & -\frac{1}{K}G_a + \Lambda *e_a - \frac{4K}{\mu(K-\nu)}D(\omega)Y_a + \frac{8K^2\nu}{\mu^2(K-\nu)^2}Y^b \wedge Y^c \wedge *e_{abc} = 0, \\ & K_{ab} = -\frac{4K\nu}{\mu(K-\nu)}\epsilon_{abc}Y^c. \end{aligned} \quad (3.13)$$

4 Background Solutions with AdS_3

In order to proceed any further in the study of a 3D quantized theory of gravity based on our model, its background solutions should be found. Towards that end, here we consider three dimensional non-Riemannian space-times of constant curvature and constant torsion[30, 31]. We also conveniently work with coordinate independent methods[6]. That is to say, we evaluate curvatures and their derivatives without differentiating any functions. The relevant differential geometric techniques are briefly explained in an appendix. Our starting point will be the structure equations satisfied by an orthonormal set of left-invariant basis 1-forms $\{e^a\}$ on AdS_3 :

$$de^a = -\frac{1}{\rho}\epsilon^a_{bc}e^b \wedge e^c. \quad (4.1)$$

Thus we determine the Levi-Civita connection 1-forms

$$\hat{\omega}^a_b = -\frac{1}{\rho}\epsilon^a_{bc}e^c, \quad (4.2)$$

and the corresponding curvature 2-forms

$$\hat{R}^a_b = -\frac{1}{\rho^2}e^a \wedge e_b. \quad (4.3)$$

Now, we set the torsion 2-forms to be

$$T^a = \frac{2}{\sigma} * e^a, \quad \sigma^2 \neq \rho^2 \iff K^a_b = -\frac{1}{\sigma}\epsilon^a_{bc}e^c. \quad (4.4)$$

Then the full curvature 2-forms turn out to be

$$R^a_b = \left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right)e^a \wedge e_b. \quad (4.5)$$

Their contractions give

$$Ric_a = 2\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right)e_a, \quad \mathcal{R} = 6\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right). \quad (4.6)$$

Substituting these in (3.4), we find

$$Q = \frac{1}{2} - \frac{\nu}{2K} - 6\nu\left(2\gamma + \frac{\beta}{2}\right)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right), \quad (4.7)$$

and in (3.6), we find

$$\lambda_a = -Q^{-1} \left(\frac{1}{\mu} + \frac{2\alpha + \beta}{\sigma} \right) \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right) e_a. \quad (4.8)$$

We must first check (3.2) for consistency:

$$\frac{1}{\nu \sigma} = Q^{-1} \left(\frac{1}{\mu} + \frac{2\alpha + \beta}{\sigma} \right) \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right). \quad (4.9)$$

Substituting for Q from (4.7), we get an algebraic consistency equation as follows:

$$2 \left(\frac{\sigma}{\mu} + 2\alpha + 4\beta + 12\gamma \right) \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right) = \frac{K - \nu}{K\nu}. \quad (4.10)$$

Next we go over to the Einstein field equations (3.8) with

$$\lambda_a = -\frac{1}{\nu \sigma} e_a \quad (4.11)$$

and organise terms to arrive at

$$(2\alpha - 2\beta - 12\gamma) \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right)^2 + \frac{1}{K} \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right) + \left(\Lambda - \frac{1}{\nu \sigma^2} \right) = 0. \quad (4.12)$$

Thus we obtain two algebraic equations (4.10) and (4.12) that are to be solved simultaneously for ρ and σ in terms of the coupling parameters $K, \Lambda, \alpha, \beta, \gamma, \mu$ and ν . In order to make further headway, we introduce new variables

$$\xi = \frac{1}{\rho}, \quad \eta = \frac{1}{\sigma}, \quad a = \frac{\mu}{4} \left(\frac{1}{\nu} - \frac{1}{K} \right) \quad (4.13)$$

and write down the following simultaneous algebraic equations:

$$\frac{1}{(\xi + \eta)} + \frac{1}{(\xi - \eta)} = \frac{1}{a^2} + \frac{(\alpha + 2\beta + 6\gamma)}{a} (\xi + \eta) + \frac{(\alpha + 2\beta + 6\gamma)}{a} (\xi - \eta), \quad (4.14)$$

$$\begin{aligned} \left(\frac{\xi + \eta}{\xi - \eta} \right) + \left(\frac{\xi - \eta}{\xi + \eta} \right) &= \left(\frac{4\nu}{K} - 1 \right) + 4\nu(2\alpha - 2\beta - 12\gamma)(\xi - \eta)(\xi + \eta) \\ &\quad + \frac{4\nu\Lambda}{(\xi - \eta)(\xi + \eta)}. \end{aligned} \quad (4.15)$$

Let us now make a further change of variables

$$t = (\xi - \eta) + (\xi + \eta), \quad s = (\xi - \eta)(\xi + \eta), \quad (4.16)$$

in terms of which we have

$$at = \frac{1}{a}s + (\alpha + 2\beta + 6\gamma)ts, \quad (4.17)$$

$$t^2 = \left(\frac{4\nu}{K} + 1\right)s + 4\nu(2\alpha - 2\beta - 12\gamma)s^2 + 4\nu\Lambda. \quad (4.18)$$

One can now solve the first equation for t in terms of s and substitute it in the second equation. Thus a quartic equation for s is reached whose solutions can be obtained in a standard way. Then working through the equations backwards, solutions for ρ and σ may be written explicitly. As they stand, they are not very instructive. However, to be concrete, we concentrate on a simpler case and discuss the parameter ranges for the existence of solutions for MMG with $\alpha = \beta = \gamma = 0$, in which case our coupled system of algebraic equations reduce to the equations of two conic sections in the $(\xi\eta)$ -plane, given by

$$\frac{(\xi - a)^2}{a^2} - \frac{\eta^2}{a^2} = 1 \quad (4.19)$$

and

$$K\left(\frac{1}{\nu} - \frac{1}{K}\right)\frac{\xi^2}{K\Lambda} + \frac{\eta^2}{K\Lambda} = 1. \quad (4.20)$$

We take $K > 0$ without loss of generality at this point, since our model is not yet coupled to matter. We also point out that solutions come in pairs with values $\eta \leftrightarrow -\eta$, as a change in sign of η means going from one orientation of the co-basis to the other, or vice versa. In what follows, we restrict attention to the cases $0 < \eta$, but extension to cases $\eta < 0$ is easy. Then we classify possible pairs (ξ, η) in accordance with the following ranges of our free parameters:

- For $\Lambda < 0$ and $-\infty < \mu < \infty$; no solution exists with $0 \leq \nu \leq K$.

- $\Lambda < 0$, $\mu > 0$, $\nu \leq 0$ or $K \leq \nu$.

There is a solution for $0 \leq \xi \leq \sqrt{\left|\frac{(K-\nu)}{\nu}K\Lambda\right|}$ and $0 < \eta < \infty$.

A second one may exist for $-\sqrt{\left|\frac{(K-\nu)}{\nu}K\Lambda\right|} \leq \xi \leq -\mu\left|\frac{(K-\nu)}{2K\nu}\right|$, depending on the magnitude of μ .

- $\Lambda < 0$, $\mu < 0$, $\nu \leq 0$ or $K \leq \nu$.

There is a solution for $\xi \leq -\sqrt{\left|\frac{(K-\nu)}{\nu}K\Lambda\right|}$ and $0 < \eta < \infty$.

A second one may exist for $\sqrt{\left|\frac{(K-\nu)}{\nu}K\Lambda\right|} \leq \xi \leq \mu\frac{(K-\nu)}{2K\nu}$, depending on the magnitude of μ .

- $\Lambda > 0, \quad \mu > 0, \quad 0 \leq \nu \leq K.$
Solutions exist for $0 \leq \eta \leq \sqrt{K\Lambda}$. Then there is a solution for $-\sqrt{\frac{K-\nu}{\nu}K\Lambda} \leq \xi \leq 0$. A second one may exist, depending on the magnitude of μ , for $\sqrt{\frac{K-\nu}{\nu}K\Lambda} \leq \xi \leq \mu \frac{(K-\nu)}{2K\nu}$.
- $\Lambda > 0, \quad \mu < 0, \quad \nu \leq 0 \text{ or } K \leq \nu.$
Two solutions exist for $\sqrt{K\Lambda} \leq \eta$ and with either $\xi \leq 0$ or $\mu \frac{(K-\nu)}{2K\nu} \leq \xi$.
- $\Lambda > 0, \quad \mu > 0, \quad \nu \leq 0 \text{ or } K \leq \nu.$
Two solutions exist for $\sqrt{K\Lambda} \leq \eta$ and with either $\xi \leq -\mu \frac{(K-\nu)}{2K\nu}$ or $0 \leq \xi$.
- $\Lambda > 0, \quad \mu < 0, \quad 0 \leq \nu \leq K.$
Solutions exist for $\sqrt{K\Lambda} \leq \eta$. One solution has $0 \leq \xi \leq \sqrt{\frac{\nu}{K-\nu}K\Lambda}$. A second one may exist, depending on the magnitude of μ , if $-\sqrt{\frac{\nu}{K-\nu}K\Lambda} \leq \xi \leq -\sqrt{|\mu| \frac{(K-\nu)}{\nu}}$.

5 Concluding Remarks

In this paper, we considered the extension of Einstein-Chern-Simons gravity (TMG) with the most general quadratic curvature terms in the action and derived the corresponding field equations by a first order constrained variational principle. We made extensive use of the concise language of exterior differential forms. Variational field equations were determined both in the absence and presence of a dynamical space-time torsion. It should be emphasised that our discussion based on the choice (2.2) of the action encompasses all currently studied models such as NMG or MMG as particular sub-cases. In order to specify the ground state of our model we then considered Riemann-Cartan space-times of constant negative curvature (i.e. AdS_3) and constant torsion as exact background solutions.

Finally we wish to add the following comments:

1. In recent literature, the generic quadratic curvature term in the action that is commonly used is given by the first alternative in Eqn.(2.5). Here we use the second alternative for technical ease and were able to present the final field equations (3.8) in a compact and geometrically transparent way.
2. The notion of 3-dimensional Riemann-Cartan spaces with constant curvature and constant torsion is not new [30, 31], but not used in this

context before. It provides a novel pathway for the construction of physically relevant 3D-gravity configurations.

3. The coupled algebraic equations (4.14) and (4.15) describe a cubic and a quartic curve, respectively, in the $(\xi\eta)$ -plane. We showed the existence of intersection points in the special case of MMG ($\alpha = \beta = \gamma = 0$) where both curves reduce to conic sections.

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7 Appendix

AdS_3 can be realized as an embedded hypersurface in a four dimensional flat space equipped with an indefinite metric $g = -dU^2 - dV^2 + dX^2 + dY^2$, written in Cartesian coordinates $\{\xi^A; A = 1, 2, 3, 4\} : (U, V, X, Y)$. In the same coordinate system, the embedding equation will be given by

$$-U^2 - V^2 + X^2 + Y^2 = -1. \quad (7.1)$$

Furthermore we know that AdS_3 is an homogeneous space with $AdS_3 = SO(2, 2)/SO(2, 1)$. In order to verify that the Lie algebra $so(2, 2)$ of isometries is a direct product of two copies of $so(2, 1)$, we consider, in our Cartesian system $\xi_A : (-U, -V, X, Y)$, the Killing vector fields J_{AB} that are given explicitly by,

$$J_{AB} = \xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A} \quad (7.2)$$

and satisfy the commutation relations:

$$[J_{AB}, J_{BC}] = \begin{cases} -J_{AC}, & \text{for } B \in \{1, 2\} \text{ and } A \neq B \neq C, \\ J_{AC}, & \text{for } B \in \{3, 4\} \text{ and } A \neq B \neq C. \end{cases} \quad (7.3)$$

It is straightforward to divide these Killing vector fields into two conjugacy classes by defining the left-invariant vector fields

$$X_0 = -J_{UV} - J_{XY}, \quad X_1 = J_{XU} + J_{YV}, \quad X_2 = J_{YU} - J_{XV}, \quad (7.4)$$

and the right-invariant vector fields

$$Y_0 = -J_{UV} + J_{XY}, \quad Y_1 = J_{XU} - J_{YV}, \quad Y_2 = -J_{YU} - J_{XV} \quad (7.5)$$

Both the left-invariant vector fields $\{X_a : a = 0, 1, 2\}$ and the right-invariant vector fields $\{Y_a : a = 0, 1, 2\}$ satisfy the same commutation relations

$$[X_0, X_1] = 2X_2, \quad [X_1, X_2] = -2X_0, \quad [X_0, X_2] = -2X_1, \quad (7.6)$$

$$[Y_0, Y_1] = 2Y_2, \quad [Y_1, Y_2] = -2Y_0, \quad [Y_0, Y_2] = -2Y_1, \quad (7.7)$$

and they commute with each other, i.e.

$$[X_a, Y_b] = 0, \quad a, b = 0, 1, 2. \quad (7.8)$$

At this point, we choose a local coordinate chart $x^\mu : (t, \chi, \theta)$ for AdS_3 such that

$$U = \cos t, V = \sin t \cosh \chi, X = \sin t \sinh \chi \cos \theta, Y = \sin t \sinh \chi \sin \theta. \quad (7.9)$$

Then we compute the following explicit expressions for $\{X_a\}$:

$$\begin{aligned} X_0 &= \cosh \chi \partial_t - \cot t \sinh \chi \partial_\chi - \partial_\theta, \\ X_1 &= -\sinh \chi \cos \theta \partial_t + (\cot t \cosh \chi \cos \theta + \sin \theta) \partial_\chi \\ &\quad + (\coth \chi \cos \theta - \cot t \operatorname{cosech} \chi \sin \theta) \partial_\theta, \\ X_2 &= -\sinh \chi \sin \theta \partial_t + (\cot t \cosh \chi \sin \theta - \cos \theta) \partial_\chi \\ &\quad + (\coth \chi \sin \theta + \cot t \operatorname{cosech} \chi \cos \theta) \partial_\theta, \end{aligned}$$

and for $\{Y_a\}$:

$$\begin{aligned} Y_0 &= \cosh \chi \partial_t - \cot t \sinh \chi \partial_\chi + \partial_\theta, \\ Y_1 &= -\sinh \chi \cos \theta \partial_t + (\cot t \cosh \chi \cos \theta - \sin \theta) \partial_\chi \\ &\quad + (-\coth \chi \cos \theta - \cot t \operatorname{cosech} \chi \sin \theta) \partial_\theta, \\ Y_2 &= \sinh \chi \sin \theta \partial_t + (-\cot t \cosh \chi \sin \theta - \cos \theta) \partial_\chi \\ &\quad + (\coth \chi \sin \theta - \cot t \operatorname{cosech} \chi \cos \theta) \partial_\theta. \end{aligned}$$

Finally, exploiting the dualities $e^b(X_a) = \delta_a^b$ and $\tilde{e}^b(Y_a) = \delta_a^b$, we determine in a unique way the following set of left-invariant co-frame 1-forms:

$$e^0 = \cosh \chi dt + \cos t \sin t \sinh \chi d\chi + \sin^2 t \sinh^2 \chi d\theta, \quad (7.10)$$

$$\begin{aligned} e^1 &= \sinh \chi \cos \theta dt + (\cos t \sin t \cosh \chi \cos \theta + \sin^2 t \sin \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \cos \theta - \cot t \sin \theta) d\theta, \end{aligned} \quad (7.11)$$

$$\begin{aligned} e^2 &= \sinh \chi \sin \theta dt + (\cos t \sin t \cosh \chi \sin \theta - \sin^2 t \cos \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \sin \theta + \cot t \cos \theta) d\theta, \end{aligned} \quad (7.12)$$

and the right-invariant co-frame 1-forms:

$$\tilde{e}^0 = \cosh \chi dt + \cos t \sin t \sinh \chi d\chi - \sin^2 t \sinh^2 \chi d\theta, \quad (7.13)$$

$$\begin{aligned} \tilde{e}^1 &= \sinh \chi \cos \theta dt + (\cos t \sin t \cosh \chi \cos \theta - \sin^2 t \sin \theta) d\chi \\ &\quad - \sin^2 t \sinh \chi (\cosh \chi \cos \theta + \cot t \sin \theta) d\theta, \end{aligned} \quad (7.14)$$

$$\begin{aligned} \tilde{e}^2 &= -\sinh \chi \sin \theta dt - (\cos t \sin t \cosh \chi \sin \theta + \sin^2 t \cos \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \sin \theta - \cot t \cos \theta) d\theta, \end{aligned} \quad (7.15)$$

It is now straightforward to verify i) that these basis 1-forms satisfy the first Cartan structure equations

$$de^a = -\epsilon^a_{bc} e^b \wedge e^c, \quad d\tilde{e}^a = -\epsilon^a_{bc} \tilde{e}^b \wedge \tilde{e}^c, \quad (7.16)$$

and that ii) in our local coordinate chart the metric tensor becomes

$$\begin{aligned} g_{AdS_3} &= -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \\ &= -\tilde{e}^0 \otimes \tilde{e}^0 + \tilde{e}^1 \otimes \tilde{e}^1 + \tilde{e}^2 \otimes \tilde{e}^2 \\ &= -dt^2 + \sin^2 t (d\chi^2 + \sinh^2 \chi d\theta^2). \end{aligned} \quad (7.17)$$

As a further remark, suppose we let the right-invariant vector fields change sign i.e. $Y_a \mapsto W_a = -Y_a$, $a = 0, 1, 2$. Note that the volume form also switches sign. Then the vector fields $\{W_a\}$ commute with the left-invariant vector fields $\{X_a\}$, but their structure constants get modified to $-2\epsilon_{abc}$. These new vector fields are explicitly written as :

$$\begin{aligned} W_0 &= -\cosh \chi \partial_t + \cot t \sinh \chi \partial_\chi - \partial_\theta, \\ W_1 &= \sinh \chi \cos \theta \partial_t - (\cot t \cosh \chi \cos \theta - \sin \theta) \partial_\chi \\ &\quad + (\coth \chi \cos \theta + \cot t \operatorname{cosech} \chi \sin \theta) \partial_\theta, \\ W_2 &= -\sinh \chi \sin \theta \partial_t + (\cot t \cosh \chi \sin \theta + \cos \theta) \partial_\chi \\ &\quad - (\coth \chi \sin \theta - \cot t \operatorname{cosech} \chi \cos \theta) \partial_\theta. \end{aligned}$$

The corresponding basis 1-forms $\{\bar{e}^a\}$ differ from the right-invariant 1-forms $\{\tilde{e}^a\}$ by an over-all minus sign:

$$\bar{e}^0 = -\cosh \chi dt - \cos t \sin t \sinh \chi d\chi + \sin^2 t \sinh^2 \chi d\theta, \quad (7.18)$$

$$\begin{aligned} \bar{e}^1 &= -\sinh \chi \cos \theta dt + (-\cos t \sin t \cosh \chi \cos \theta + \sin^2 t \sin \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \cos \theta + \cot t \sin \theta) d\theta, \end{aligned} \quad (7.19)$$

$$\begin{aligned} \bar{e}^2 &= \sinh \chi \sin \theta dt + (\cos t \sin t \cosh \chi \sin \theta + \sin^2 t \cos \theta) d\chi \\ &\quad - \sin^2 t \sinh \chi (\cosh \chi \sin \theta - \cot t \cos \theta) d\theta, \end{aligned} \quad (7.20)$$

and satisfy the following structure equations:

$$d\bar{e}^a = \epsilon^a_{bc} \bar{e}^b \wedge \bar{e}^c. \quad (7.21)$$

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